

The untilted diffuse matter Bianchi V Universe

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Abstract

A diffuse matter filled Type V Universe is studied. The anisotropic behaviour, the distortion caused to the CMBR and the parameter region allowed by present cosmological bounds are examined. It is shown how the overall sky pattern of temperature anisotropies changes under a non-infinitesimal spatial coordinate transformation that preserves the Type V manifest homogeneity.

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I. INTRODUCTION

The general solution of Type V spatially homogenous cosmology in the realistic case of untilted diffuse mater $p = g \rho$, [1], [2], [3] is studied in various forms of the metric. This kind of equation of state covers the actual matter composition of the Universe for most of it's time evolution. It covers both the radiation era and the dust epoch. Note also, that not only theoretical considerations (Mach's principle) but also observational data, suggest a value close to zero for the rotation of the Universe [4], thus validating our assumption about untilted fluid. This solution describes an open Universe, a feature that is lately favorable [5], and at the same time is not trivially simple like the Kasner family.

The existence of a primordial global anisotropy in our Universe is an open issue. The promising string theories suggest an anisotropic expansion that compactified all but four dimensions. It is natural to expect that the remaining dimensions unfold to one of the nine known types of spatially homogenous models, most of which are anisotropic.

On the other hand it is well known that the smoothness of the cosmic microwave background radiation (CMBR) suggests that the Friedmann-Robertson-Walker (FRW) cosmological model is a fair description of the actual present Universe. However the radiation spectra has small fluctuations originating, in general, from pertubations produced by some physical mechanism (inflation-topological defects) and from primordial anisotropies and inhomogeneities. If the Universe experienced an inflationary period, any primordial overall anisotropy may have been smoothed but not erased. A possible realistic scenario then, could be that we had, after the Planck regime, an anisotropic Universe that appeared more isotropic after inflation and almost FRW during the last scattering and afterwards. It is certainly important to know more about the *characteristics* and the *evolution* of this global geometric primordial anisotropy.

The above mentioned scenario can be fully studied, only if we study the general solutions of the nine types of spatially homogeneous cosmologies. Interesting works have appeared in the literature regarding the connection of the various Bianchi types with the observed

cosmological data. There have been works assuming small anisotropic perturbations to the FRW solution [4], [6], [7] which are valid after last scattering and others [8], [9], which incorporate the exact solutions for various types of homogeneous models. All these works study the final temperature patterns induced on the CMBR. The basic conclusion is that, if someone wants the distortion of the null geodesics after last scattering to be small in order to comply with the observed quadrupole, then one has to fix the parameters of the various models to produce a metric very close to the FRW. Consequently, they set upper limits on the relative shear $\left(\frac{\Sigma}{\Theta}\right)_0$, where Σ is the shear anisotropy and $\Theta = 3H$ is the expansion rate, and on the vorticity.

There is also the important work of W. R. Stoeger, R. Maartens, and G. F. R. Ellis [10], [11] in which a theorem is proven stating that if all observers measured CMBR to be almost isotropic, in our past light cone from the last scattering till now, then this universe has to be almost FRW in this region. They are referring to an expanding Universe with noninteracting matter and radiation. Then based on this theorem and some not easily testable simplifications they set upper limits on the allowed relative shear $\left(\frac{\Sigma}{\Theta}\right)_0$.

One can in principle use the CMBR data in order to determine the allowed parameter space and the type of the various homogenous cosmological solutions. Assuming that the actual Universe after last scattering is a dust Bianchi V one, we studied the distortion of the initial isotropic temperature pattern caused by the null geodesics. We present a method that yields simple system of equations for the null geodesics (a similar idea has been used in [12]). We found that for some parameters that make the solution close to the FRW one, it is possible to have for the period from last scattering till now very small quadrupole anisotropy compatible with the present data. For parameters not close to the FRW solution we always failed to find acceptably small quadrupole anisotropy. The most important feature we found is that a non-infinitesimal and not orthogonal transformation of coordinates that preserves the Bianchi V manifest homogeneity of the spacetime, leads to a non trivial transformation of the repeatedly appearing in the literature overall sky-pattern of anisotropies, although does not affect the covariant properties carried in that pattern.

The paper is organized as follows: In section II the Einstein equations are setup and completely solved. We present the solution in the off diagonal form, in order to make manifest the dependency on the spatial coordinate system, of the calculations leading to the temperature fluctuations as a function of the sky angles. In section III the anisotropic behaviour and its effect on the temperature pattern, meaning the plot of the temperature as a function of the sky angles, is studied. The magnitude of the pattern distortion resulted by the use of the non-diagonal form of the metric is presented. Some concluding remarks are also given.

II. THE UNTILTED DIFFUSE MATTER BIANCHI V COSMOLOGY

Einstein equations are written as

$${}^{(4)}R_{AB} - \frac{1}{2} {}^{(4)}R g_{AB} = \kappa T_{AB}, \quad (1)$$

where $\kappa = -8\pi G$ the Einstein constant and T_{AB} the energy-momentum tensor. Capital Latin indices are spacetime indices $A, B, .. = 0, ..., 3$ while lower Latin are spatial indices $i, j, .. = 1, 2, 3$.

In the 3+1 analysis, we use the lapse and shift functions in order to write the 4-metric as

$$ds^2 = (N_i N^i - N^2) dt^2 + 2N_i dx^i dt + g_{ij} dx^i dx^j. \quad (2)$$

Then, Einstein equations (1) are equivalent to the following system :

$$K_{i|j}^j - K_{|i} = -\kappa N T_i^0 \quad (3)$$

$$K_j^i K_i^j - K^2 + {}^{(3)}R = -2\kappa (T_0^0 - T_i^0 N^i) \quad (4)$$

$$\begin{aligned} & \dot{K}_j^i - N K K_j^i + N {}^{(3)}R_j^i + g^{il} N_{|j}l - (K_{j|l}^i N^l + K_l^i N_{|j}^l - K_j^l N_{|l}^i) \\ &= \kappa N (T_j^i + T_j^0 N^i) - \frac{N}{2} (\kappa {}^{(4)}T) \delta_j^i \end{aligned} \quad (5)$$

The extrinsic curvature K_{ij} is given by

$$K_{ij} = \frac{1}{2N} \left(N_{i|j} + N_{j|i} - \frac{\partial g_{ij}}{\partial t} \right), \quad (6)$$

while

$$K_j^i \equiv g^{il} K_{lj}, \quad K \equiv K_i^i, \quad (7)$$

where g^{ij} is the inverse of the 3-metric g_{ij} . We also define

$${}^{(3)}R_j^i \equiv g^{il} {}^{(3)}R_{lj}, \quad T_B^A \equiv g^{AC} T_{CB}, \quad {}^{(4)}T \equiv T_A^A. \quad (8)$$

The class of spatially homogeneous spacetimes is characterised by the existence of an m -dimensional isometry group of motions, acting on each surface of simultaneity Σ_t . If $m > 3$ and there is no proper invariant subgroup of dimension 3, the spacetime is of the Kantowski-Sachs type [13], and will not concern us further. When $m = 3$, the dimension of Σ_t , there exist three basis one-forms σ^a satisfying

$$d\sigma^\alpha = C_{\beta\gamma}^\alpha \sigma^\beta \wedge \sigma^\gamma \Leftrightarrow \sigma_{i,j}^\alpha - \sigma_{j,i}^\alpha = 2C_{\beta\gamma}^\alpha \sigma_i^\gamma \sigma_j^\beta, \quad (9)$$

where $C_{\beta\gamma}^\alpha$ are the structure constants of the corresponding isometry group. Greek indices number the different basis 1-forms and take values in the range $1, \dots, 3$. In this case there are coordinates t, x^i such that the line element 2 takes the form

$$ds^2 = \left[N_\alpha(t) N^\alpha(t) - N^2(t) \right] dt^2 + 2N_\alpha(t) \sigma_i^\alpha(x) dx^i dt + \gamma_{\alpha\beta}(t) \sigma_i^\alpha(x) \sigma_j^\beta(x) dx^i dx^j. \quad (10)$$

Inserting the line element Eq.(10) into the system of equations Eq.(3), Eq.(4), Eq.(5) we get, via repeated use of Eq.(9) the following set of ordinary differential equations for the Bianchi-type spatially homogeneous spacetimes:

$$K_\nu^\mu C_{\alpha\mu}^\nu - K_\alpha^\mu C_{\mu\nu}^\nu = \frac{\kappa N}{2} T_\alpha^0, \quad (11)$$

$$K_\beta^\alpha K_\alpha^\beta - K^2 + {}^{(3)}R + 2\Lambda = -2\kappa \left(T_0^0 - T_\rho^0 N^\rho \right) \quad (12)$$

$$\dot{K}_\beta^\alpha - N K K_\beta^\alpha + N R_\beta^\alpha + 2N^\rho \left(K_\mu^\alpha C_{\beta\rho}^\mu - K_\beta^\mu C_{\mu\rho}^\alpha \right) = \kappa N \left(T_\beta^\alpha + T_\beta^0 N^\alpha \right) - \frac{N}{2} \left[\kappa (T_0^0 + T_\rho^0) \right] \delta_\beta^\alpha \quad (13)$$

where

$$K_{\beta}^{\alpha} \equiv \gamma^{\alpha\rho} K_{\rho\beta}, \quad R_{\beta}^{\alpha} \equiv \gamma^{\alpha\rho} R_{\rho\beta} \quad \text{and} \quad (14)$$

$$K_{\alpha\beta} = -\frac{1}{2N} \left(\dot{\gamma}_{\alpha\beta} + 2\gamma_{\alpha\nu} C_{\beta\rho}^{\nu} N^{\rho} + 2\gamma_{\beta\nu} C_{\alpha\rho}^{\nu} N^{\rho} \right), \quad (15)$$

$$R_{\alpha\beta} = C_{\sigma\tau}^{\kappa} C_{\mu\nu}^{\lambda} \gamma_{\alpha\kappa} \gamma_{\beta\lambda} \gamma^{\sigma\nu} \gamma^{\tau\mu} + 2C_{\alpha\kappa}^{\lambda} C_{\beta\lambda}^{\kappa} + 2C_{\alpha\kappa}^{\mu} C_{\beta\lambda}^{\nu} \gamma_{\mu\nu} \gamma^{\kappa\lambda} + 2C_{\beta\kappa}^{\lambda} C_{\mu\nu}^{\mu} \gamma_{\alpha\lambda} \gamma^{\kappa\nu} + 2C_{\alpha\kappa}^{\lambda} C_{\mu\nu}^{\mu} \gamma_{\beta\lambda} \gamma^{\kappa\nu}. \quad (16)$$

The components of the energy-momentum tensor appearing in the system of Eq.(11), Eq.(12), Eq.(13), are the time-dependent parts of the corresponding full components T_B^A . The energy-momentum tensor of a perfect fluid can be written as

$$T^{AB} = (\rho + p)u^A u^B + pg^{AB}, \quad u^A u_A = -1, \quad (17)$$

where u^A is its four-velocity ($u_A \equiv g_{AB}u^B$), ρ is the total energy density and p the pressure.

The conservation equations $T^{AB}{}_{;B} = 0$ are equivalent to

$$\dot{\rho} + \Theta(\rho + p) = 0 \quad (18)$$

$$(\rho + p) \dot{u}^A + p_{;B} g^{AB} + \dot{p} u^A = 0, \quad (19)$$

where $\Theta = -K = u^A_{;A}$ is the expansion scalar and the dot denotes ∇_u . Now the last equations and the system Eq. (3), Eq.(4), Eq.(5) are integrable given an equation of state $p = p(\rho)$. A very general equation of state is that of the diffuse matter

$$p = g\rho, \quad |g| \leq 1 \quad (20)$$

with g a constant. It includes the dust case for $g = 0$, gravitational waves ($g = 1/3$), cosmic string networks ($g = -1/3$), domain walls ($g = -2/3$) and free massless scalar fields ($g = +1$).

In spatially homogeneous spacetimes, if the perfect fluid moves orthogonal to the spatial slices, we have

$$u^A = \frac{1}{N} \begin{pmatrix} 1 & 0 \end{pmatrix} \quad (21)$$

in the so called synchronous coordinate frame ($N^a = 0$) which in our case it is also commoving. Using now Eq.(15) we can write Eq.(18) as follows :

$$\frac{\dot{\rho}}{\rho + p} + \frac{\dot{\gamma}}{2\gamma} = 0, \quad (22)$$

where $\gamma \equiv \det(\gamma_{\alpha\beta})$. The dot denotes ordinary differentiation with respect to the time label that characterises the slices Σ_t . Integrating the above expression with the help of Eq. (20) we find

$$\rho = \rho_1 \gamma^{-\frac{g+1}{2}}. \quad (23)$$

The quantity ρ_1 is a positive constant. Note that Eq.(19) becomes an identity since $p = p(t)$ and u^A is a geodesic field. The fact that our perfect fluid is taken to be untilted leads to the simplification $T_\alpha^0 = 0$ and thus the linear equations Eq.(11) coincide with the vacuum ones, i.e. they are (when N^α vanishes)

$$\gamma^{\mu\beta} C_{\alpha\mu}^\nu \dot{\gamma}_{\nu\beta} - \gamma^{\mu\beta} C_{\mu\nu}^\nu \dot{\gamma}_{\alpha\beta} = 0. \quad (24)$$

We now *specialise* to the case of the Bianchi type V, i.e. we set $C_{13}^1 = -C_{31}^1 = C_{23}^2 = -C_{32}^2 = 1/2$, all other vanish. It can be shown with the aid of a time independent automorphism inducing diffeomorphism (A.I.D.) [14], that we can choose, keeping the full generality, $\gamma_{13} = \gamma_{23} = 0$, $\gamma_{11}\gamma_{22} - (\gamma_{12})^2 = (\gamma_{33})^2$. It can further be shown that, after completely integrating Einstein equations at this stage, the off diagonal element can be rotated away through a transformation $\bar{\gamma}_{\alpha\beta} = \Lambda_\alpha^\mu \Lambda_\beta^\nu \gamma_{\mu\nu}$ with Λ a constant automorphism matrix of the form

$$\Lambda = \begin{pmatrix} r & s & 0 \\ t & u & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (25)$$

Such a transformation emerges as the effect on $\gamma_{\alpha\beta}$ of a time independent "frozen" A.I.D., see Appendix of [15]; Thus the appearance of a non vanishing γ_{12} is tantamount to the use of a spatial coordinate system different from the system supporting the diagonal form

of the solutions. In what follows we keep the non diagonal element non zero in order to solve the geodesic equations in this different spatial coordinate choice and thus be able to discuss the dependency of the temperature pattern on this choice. One would wonder why we should make the endeavour to solve the Einstein equations in this off diagonal form and not just take the diagonal solution and obtain a non diagonal form through an arbitrary time dependent automorphism. The reason is that only the constant automorphism of the form discussed above, keep us within the space of solutions, as it can be proven using the off diagonal solution presented here. This off diagonal solution represents the full space of solutions (with $\gamma_{13} = \gamma_{23} = 0$ of course). The metric $\gamma_{\alpha\beta}$ is therefore taken to be

$$\gamma_{\alpha\beta} = \begin{pmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & f \end{pmatrix}, \quad \gamma = f^3 \Leftrightarrow ac - b^2 = f^2. \quad (26)$$

and Eqs. (24) are then identically satisfied. An off-diagonal matrix of the above form gives the following Ricci tensor Eq.(16) and Ricci scalar :

$$R_{\alpha\beta} = 2\gamma^{-1/3}\gamma_{\alpha\beta}, \quad {}^{(3)}R = 6\gamma^{-1/3}. \quad (27)$$

As far as time is concerned , we adopt the gauge fixing condition $N = \sqrt{\gamma}$, since this simplifies the form of the equations. Consequently, the remaining Einstein equations Eq.(12), Eq.(13) read

$$\dot{\gamma}^{\alpha\beta}\dot{\gamma}_{\alpha\beta} + \left(\frac{\dot{\gamma}}{\gamma}\right)^2 - 24\gamma^{2/3} + 8\gamma\kappa\rho = 0 \quad (28)$$

$$\ddot{\gamma}_{\alpha\beta} - \gamma^{\mu\nu} \dot{\gamma}_{\alpha\mu} \dot{\gamma}_{\beta\nu} + G(\gamma)\gamma_{\alpha\beta} = 0 \quad (29)$$

with

$$G(\gamma) \equiv -4\gamma^{2/3} + \gamma\kappa(\rho - p). \quad (30)$$

Taking the trace of Eq.(29), one arrives at

$$\left(\frac{\dot{\gamma}}{\gamma}\right)' + 3G(\gamma) = 0 \quad (31)$$

which has a first integral

$$w \equiv \left(\frac{\dot{\gamma}}{\gamma} \right)^2 + U(\gamma) = \text{const}, \quad (32)$$

where

$$U(\gamma) \equiv 6 \int \frac{G(\gamma)}{\gamma} d\gamma. \quad (33)$$

We make the conformal transformation

$$\varpi_{\alpha\beta} \equiv \gamma^{-1/3} \gamma_{\alpha\beta} \quad (34)$$

and then, due to Eq.(27) and Eq.(31) we can write Eq.(29) as

$$\ddot{\varpi}_{\alpha\beta} - \varpi^{\mu\nu} \dot{\varpi}_{\alpha\mu} \dot{\varpi}_{\beta\nu} = 0. \quad (35)$$

The above equation can be integrated immediately , giving

$$\dot{\varpi}_{\alpha\beta} = \vartheta_\alpha^\delta \varpi_{\delta\beta}. \quad (36)$$

Since $\varpi \equiv \det(\varpi_{\alpha\beta}) = 1 \Rightarrow \vartheta_\alpha^\alpha = 0$. The system of Eq.(36) is equivalent to the following set of equations

$$(\gamma^{-1/3}a)^\cdot = \vartheta_1^1(\gamma^{-1/3}a) + \vartheta_1^2(\gamma^{-1/3}b) \quad (37)$$

$$(\gamma^{-1/3}b)^\cdot = \vartheta_1^1(\gamma^{-1/3}b) + \vartheta_1^2(\gamma^{-1/3}c) \quad (38)$$

$$(\gamma^{-1/3}b)^\cdot = \vartheta_2^1(\gamma^{-1/3}a) + \vartheta_2^2(\gamma^{-1/3}b) \quad (39)$$

$$(\gamma^{-1/3}c)^\cdot = \vartheta_2^1(\gamma^{-1/3}b) + \vartheta_2^2(\gamma^{-1/3}c) \quad (40)$$

and also the relations $\vartheta_1^3 = \vartheta_2^3 = \vartheta_3^1 = \vartheta_3^2 = \vartheta_3^3 = 0$. Then $\vartheta_2^2 = -\vartheta_1^1$. At this stage we could invoke a constant automorphism matrix of the form earlier given and set the matrix ϑ_β^α to the form

$$\vartheta = \begin{pmatrix} \vartheta_1^1 & 0 & 0 \\ 0 & -\vartheta_1^1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (41)$$

which would directly led us to the diagonal form of the solution [1], [3], [2] (or Joseph's [16], in the vacuum case). However since our aim is to uncover the dependency of the temperature pattern on the spatial coordinate system, we choose to keep the non essential parameters in the game. The quadratic Einstein equation Eq.(28) and the system Eq.(37) - Eq.(40) imply

$$(\vartheta_1^1)^2 + \vartheta_1^2 \vartheta_2^1 = \frac{w}{3} - \frac{1}{3}U(\gamma) - 12\gamma^{2/3} + 4\gamma\kappa\rho, \quad (42)$$

where the function $U(\gamma)$ for the diffuse matter equation of state Eq.(20) is written as

$$U(\gamma) = -36\gamma^{2/3} + 12\kappa\rho_1\gamma^{\frac{1-g}{2}}. \quad (43)$$

Then Eq.(42) reduces the independent constants :

$$\frac{w}{3} = (\vartheta_1^1)^2 + \vartheta_1^2 \vartheta_2^1. \quad (44)$$

For $b \neq 0$, $\vartheta_2^1 \vartheta_1^2 \neq 0$, we have from Eq.(38), Eq.(39) and Eq.(26) that

$$\vartheta_2^1 a^2 - 2\vartheta_1^1 ab - \vartheta_1^2 b^2 = \vartheta_1^2 \gamma^{2/3}. \quad (45)$$

Since Eq.(40) can be derived from the other three of the system Eq.(37) - Eq.(40), it suffices to solve the system of Eq. (37), Eq.(39) and Eq.(45). The solution of the system Eq. (37), Eq.(39) gives the expressions $\gamma^{-1/3}a$, $\gamma^{-1/3}b$ as sums of two exponential functions of time. Only for $w > 0$ these solutions satisfy also Eq.(45). Thus finally, the general solution of the Bianchi V universe with an orthogonal perfect fluid and linear equation of state, in the off diagonal form, is given as follows :

$$\begin{aligned} a(t) &= \gamma^{1/3}(t) \left[k \exp \left(\sqrt{\frac{w}{3}} t \right) + \frac{3(\vartheta_1^2)^2}{4wk} \exp \left(-\sqrt{\frac{w}{3}} t \right) \right] \\ b(t) &= \gamma^{1/3}(t) \left[\frac{k}{\vartheta_1^2} \left(\sqrt{\frac{w}{3}} - \vartheta_1^1 \right) \exp \left(\sqrt{\frac{w}{3}} t \right) - \frac{3\vartheta_1^2}{4wk} \left(\sqrt{\frac{w}{3}} + \vartheta_1^1 \right) \exp \left(-\sqrt{\frac{w}{3}} t \right) \right] \\ c(t) &= \gamma^{1/3}(t) \left[\frac{k}{(\vartheta_1^2)^2} \left(\sqrt{\frac{w}{3}} - \vartheta_1^1 \right)^2 \exp \left(\sqrt{\frac{w}{3}} t \right) + \frac{3}{4wk} \left(\sqrt{\frac{w}{3}} + \vartheta_1^1 \right)^2 \exp \left(-\sqrt{\frac{w}{3}} t \right) \right] \\ f(t) &= \gamma^{1/3}(t) = \left[h^{-1} \left(\frac{t}{6} \right) \right]^2, \quad h(x) = \int \frac{dx}{x \sqrt{w + 36x^4 - 12\kappa\rho_1 x^3}}. \end{aligned} \quad (46)$$

The solution contains five arbitrary constants k , ϑ_1^1 , ϑ_1^2 , ϑ_2^1 , ρ_1 . The integration constant that arises in the expression giving $\gamma(t)$ is absorbable (non-essential) and is fixed by choosing

the zero point on the time axis. One can further realise that three of the five constants are also absorbable as explained above, leaving us with two essential parameters one for the matter and one for the gravity, as expected. The counting of the number of essential parameters for the vacuum Bianchi cosmologies can be understood as follows: From the initial twelve constants $\gamma_{\alpha\beta}(t), \dot{\gamma}_{\alpha\beta}(t_o)$ we subtract twice the number of independent first class constraints and the number of outer automorphism parameters. In our Type V case this reach $12 - 2 \times 4 - 3 = 1$ essential parameter. For Type II we have $12 - 2 \times 3 - 4 = 2$ and likewise for other Types.

If one employs the above derived solution to describe the dust filled Universe, after last scattering, it is possible to constraint some of the free parameters. Thus

$$w = \frac{144}{H^4} \frac{q - \frac{\Omega}{2}}{\left(2 - q - \frac{3\Omega}{2}\right)^3}, \quad (47)$$

$$\rho_1 = \frac{-1}{\kappa} \frac{3\sqrt{8}\Omega}{H \left(2 - q - \frac{3\Omega}{2}\right)^{3/2}}, \quad (48)$$

$$\gamma = \frac{1}{H^6} \frac{8}{\left(2 - q - \frac{3\Omega}{2}\right)^3}, \quad (49)$$

where q is the deceleration parameter, H is the Hubble rate and Ω is the flatness parameter. In their definition the mean scale factor γ , has been used. The above expressions will be used to calculate the constants $w, \rho_1, \gamma_{today} = \gamma_o$ from the measurable H_o, q_o, Ω_o .

The above relations impose the following constraints on the allowed parameter space of the solution:

$$2 - q - \frac{3\Omega}{2} > 0, \quad q - \frac{\Omega}{2} > 0. \quad (50)$$

The above constraints imply that $0 < q < 2$ and $0 < \Omega < 1$.

III. THE ANISOTROPIC BEHAVIOR

It is well known [17] that, under the assumptions of zero cosmological constant, the validity of the dominant energy condition and the positiveness of pressure, Bianchi types I,

V and VII are the only ones that can satisfy four isotropisation conditions enlisted below. Of course this by no means implies that the solution will indeed isotropise. We have to know the exact solutions in order to examine the anisotropic behaviour. These four conditions of C. B. Collins, S. W. Hawking, are as follows :

- (i) $\gamma \rightarrow \infty$
- (ii) $T^{00} > 0$ and $\frac{T^{0i}}{T^{00}} \rightarrow 0$
- (iii) $\frac{\Sigma}{\Theta} \rightarrow 0$, where $2\Sigma^2 \equiv \sigma^{AB}\sigma_{AB}$
- (iv) $\gamma^{-1/3}\gamma_{\alpha\beta} \rightarrow \text{const}$

The limits are understood to be taken as the proper time approach infinity $\tau \rightarrow +\infty$. Referring to the first condition let us study the behaviour of the quantity $\gamma(\tau)$, for the metric presented in the previous section. The proper time τ is given, in the dust case and for an expanding universe, by the following relation

$$\tau(x) = \int_0^x F(x)dx, \quad F(x) \equiv \frac{6x^2}{\sqrt{w - 12\kappa\rho_1 x^3 + 36x^4}}, \quad (51)$$

where $x \equiv \gamma^{1/6}$ and the choice of the origin of time is $\tau = 0$ for $\gamma = 0$. It is easily verified that $\gamma \rightarrow \infty$ for $\tau \rightarrow +\infty$. The second condition is trivially satisfied due to the orthogonality of the four velocity u^A . For the third criterion, we can compute the shear anisotropy Σ via the shear tensor σ^{AB} in the basis where $N^\alpha = 0$

$$2\Sigma^2 = K^\alpha_\beta K^\beta_\alpha - \frac{1}{3}K^2 = \frac{1}{6N^2} \left(\frac{\dot{\gamma}}{\gamma} \right)^2 - \left({}^{(3)}R + 2\kappa T^0_0 \right). \quad (52)$$

Finally, we find for the solution Eq.(46)

$$\frac{\Sigma}{\Theta} = \left[\frac{1}{3} \frac{w}{w - U(\gamma)} \right]^{1/2}. \quad (53)$$

One should note that for $\gamma \rightarrow 0$ we have $\frac{\Sigma}{\Theta} \rightarrow \frac{1}{\sqrt{3}}$. Furthermore, the function $\frac{\Sigma}{\Theta}$ is a strictly decreasing function of γ and $\lim_{\gamma \rightarrow \infty} \frac{\Sigma}{\Theta} = 0$. The last condition (iv), holds for dust, since we can prove that $\lim_{x \rightarrow +\infty} t(x) < +\infty$. Therefore the solution satisfies all the specified

conditions. In the following, we will examine some more criteria, that can characterise the behaviour of a homogeneous model.

A cosmological model with a perfect fluid source is said, according to GFR Ellis [18], to be close to a FRW model in some open set of the spacetime, if for some suitably small constant $\varepsilon \ll 1$ the following inequalities hold, in this open set:

$$\frac{\Sigma}{\Theta} < \varepsilon, \quad \frac{\sqrt{|E^{\mu\nu} E_{\mu\nu}|}}{H^2} < \varepsilon, \quad \frac{\sqrt{|H^{\mu\nu} H_{\mu\nu}|}}{H^2} < \varepsilon, \quad (54)$$

where $E_{\lambda\nu} \equiv C_{\lambda\mu\nu\kappa} \eta^\mu \eta^\kappa$, $H_{\lambda\nu} \equiv \tilde{C}_{\lambda\mu\nu\kappa} \eta^\mu \eta^\kappa$ are the electric and magnetic parts of the Weyl tensor $C_{\lambda\mu\nu\kappa}$ respectively. The dual Weyl tensor $\tilde{C}_{\lambda\mu\nu\kappa}$ is defined as $\tilde{C}_{\lambda\mu\nu\kappa} \equiv \frac{1}{2} \eta_{ab}^{st} C_{stcd}$ with η^{abcd} the complete antisymmetric tensor and $\eta^{1234} \equiv \frac{1}{\sqrt{-g}}$. These covariant criteria are believed to indicate that the model under consideration is close to a FRW one. For the solution considered here we find that

$$\frac{\sqrt{|E^{\mu\nu} E_{\mu\nu}|}}{H^2} = \sqrt{6w} \frac{\sqrt{|2w + 3U|}}{w - U}, \quad \frac{\sqrt{|H^{\mu\nu} H_{\mu\nu}|}}{H^2} = 6\sqrt{6w} \frac{\gamma^{1/3}}{w - U}. \quad (55)$$

These inequalities always hold in some interval (γ_1, γ_2) if we choose a sufficient small value for the parameter w .

Finally the distortion that the geometric anisotropies induce to the temperature pattern via the null geodesics will be examined. Through this analysis it will be possible to find also the parameter region that is allowed from the current CMBR constraints. The small anisotropies of the CMBR are studied computing the angular variation of the microwave temperature, $T_o(\theta, \varphi)$ characterising the frequency distribution of the radiation. One defines

$$\frac{\Delta T}{T}(\theta, \varphi) \equiv \frac{T_o(\theta, \varphi) - \langle T_o \rangle}{\langle T_o \rangle}, \quad (56)$$

where $\langle T_o \rangle$ is the mean, over the sky, observed temperature. The above field is decomposed in spherical harmonics

$$\frac{\Delta T}{T}(\theta, \varphi) = \sum_{l=1}^{\infty} \sum_{m=-l}^l a_{lm} Y_{lm}(\theta, \varphi). \quad (57)$$

The various multipole moments a_{lm} are defined by

$$a_{lm} = \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta \frac{\Delta T}{T} Y_{l,-m}(\theta, \varphi) \quad (58)$$

and we can finally define

$$(a_l)^2 \equiv \langle |a_{lm}|^2 \rangle = \frac{1}{4\pi} \sum_{m=-l}^l |a_{lm}|^2. \quad (59)$$

After subtracting the dipole moment of anisotropy which is ascribed to the local motion of our sun and our galaxy around the local supercluster, one finds a small anisotropy of the order of 10^{-5} . Part of the remaining anisotropy is the quadrupole fluctuation.

The cosmological redshift z_e of a photon emitted at last scattering can be expressed as [19]

$$1 + z_e = \frac{(u_A k^A)_e}{(u_A k^A)_o}, \quad (60)$$

where k^A is the tangent vector to the null geodesics and u^A is the four velocity which in our case is vertical to the homogenous slices. The subscripts denote the emission time and the present time. It is obvious that

$$u_A k^A = -N(t)k^0. \quad (61)$$

A unit vector, tangent to the spatial homogenous slice today t_o , in our position O , can be written as

$$e^A \equiv (u^A)_o + \frac{(k^A)_o}{(u_A k^A)_o} = \sin \theta \cos \varphi Y_1 + \sin \theta \sin \varphi Y_2 + \cos \theta Y_3, \quad (62)$$

where the sky angles θ, φ specify the direction of the incoming photon. Note that we introduced on the spatial hypersurface an orthonormal basis $\{Y_i\}$ with $g(Y_i, Y_j) = \delta_{ij}$. It is related to the invariant basis $\{\sigma^\alpha\}$:

$$\sigma^1 = e^{-x^1} dx^2, \quad \sigma^2 = e^{-x^1} dx^3, \quad \sigma^3 = dx^1. \quad (63)$$

$$X_1 = e^{-x^1} \partial_2, \quad X_2 = e^{-x^1} \partial_3, \quad X_3 = \partial_1. \quad (64)$$

by the following relations

$$Y_1 = \frac{1}{\sqrt{a_o}} X_1, \quad Y_2 = \frac{1}{\sqrt{f_o}} X_3, \quad Y_3 = \frac{\sqrt{a_o}}{f_o} \left(X_2 - \frac{b_o}{a_o} X_1 \right).$$

where $\partial_i \equiv \partial/\partial x^i$. The symbols a_o , b_o , f_o denote the scale factors evaluated at present. We can parametrise our position without losing the generality demanding $(x^i)_o = 0$. Then we get

$$Y_1 = \frac{1}{\sqrt{a_o}} \partial_2, \quad Y_2 = \frac{1}{\sqrt{f_o}} \partial_1, \quad Y_3 = \frac{\sqrt{a_o}}{f_o} \left(\partial_3 - \frac{b_o}{a_o} \partial_2 \right).$$

Finally we can express the present components of the tangent to the null geodesics vector from Eq.(60), Eq.(61)

$$\begin{aligned} (k^1)_o &= -(k^0)_o \sqrt{\frac{\gamma_o}{f_o}} \sin \theta \sin \varphi, \\ (k^2)_o &= -(k^0)_o \sqrt{\frac{\gamma_o}{a_o}} \left(\sin \theta \cos \varphi - \frac{b_o}{a_o} \cos \theta \right), \\ (k^3)_o &= -(k^0)_o \frac{\sqrt{\gamma_o a_o}}{f_o} \cos \theta. \end{aligned} \tag{65}$$

Our purpose is to find the distortion that the anisotropy of the homogenous spatial slices induce, in the propagation of the photons and consequently in $z_e(\theta, \varphi)$. It is well known that the inner product of a geodesic field k^A with a Killing vector field ξ^A is constant along the geodesics. In our case, there are three Killing vectors tangent to the homogenous hypersurfaces [20],

$$\xi_1 = \partial_2, \quad \xi_2 = \partial_3, \quad \xi_3 = \partial_1 + x^2 \partial_2 + x^3 \partial_3 \tag{66}$$

The three constant quantities, along the geodesics, $K_\alpha = g_{AB} \xi_\alpha^A k^B$ can be used to determine the null vector as

$$k^1 = \gamma^{-1/3} (K_3 - K_1 x^2 - K_2 x^3) \tag{67}$$

$$k^2 = \gamma^{-2/3} e^{2x^1} (K_1 \gamma_{22} - K_2 \gamma_{12}) \tag{68}$$

$$k^3 = \gamma^{-2/3} e^{2x^1} (K_2 \gamma_{11} - K_1 \gamma_{12}). \tag{69}$$

From Eq.(65) we can compute the three constants

$$K_\alpha = - \left(k^0 \right)_o \Lambda_\alpha , \quad (70)$$

$$\Lambda_1 = \frac{b_o}{c_o} \Lambda_2 + \frac{\gamma_o^{7/6}}{c_o \sqrt{a_o}} \left(\sin \theta \cos \varphi - b_o \gamma_o^{-1/3} \cos \theta \right) , \quad (71)$$

$$\Lambda_2 = c_o \left[\sqrt{a_o} \gamma_o^{1/6} \cos \theta + \sqrt{\frac{\gamma_o}{a_o}} \frac{b_o}{c_o} \left(\sin \theta \cos \varphi - b_o \gamma_o^{-1/3} \cos \theta \right) \right] , \quad (72)$$

$$\Lambda_3 = \gamma_o^{2/3} \sin \theta \sin \varphi . \quad (73)$$

If ζ is an affine parameter along the null geodesics then $k^A = dx^A/d\zeta$. The last relation and $ds^2 = 0$, give using Eq.(70), and Eqs.(67-69).

$$k^0 = \frac{(k^0)_o}{\sqrt{\gamma}} \Xi(\theta, \varphi, t_o; t) \quad (74)$$

$$\Xi(\theta, \varphi, t_o; t) \equiv \left[\gamma^{-1/3} \left(\Lambda_3 - \Lambda_1 x^2 - \Lambda_2 x^3 \right)^2 + e^{2x^1} \gamma^{-2/3} \left(\Lambda_1^2 \gamma_{22} + \Lambda_2^2 \gamma_{11} - 2\Lambda_1 \Lambda_2 \gamma_{12} \right) \right]^{1/2} . \quad (75)$$

Finally, we find from Eq.(74) and Eqs.(67-69) that the x^i along the geodesics are given by

$$\frac{dx^1}{dt} = -\gamma(t)^{1/6} \frac{1}{\Xi(t)} \left(\Lambda_3 - \Lambda_1 x^2 - \Lambda_2 x^3 \right) \quad (76)$$

$$\frac{dx^2}{dt} = -\gamma(t)^{-1/6} \frac{1}{\Xi(t)} e^{2x^1} [\Lambda_1 c(t) - \Lambda_2 b(t)] \quad (77)$$

$$\frac{dx^3}{dt} = -\gamma(t)^{-1/6} e^{2x^1} [\Lambda_2 a(t) - \Lambda_1 b(t)] . \quad (78)$$

This system of equations together with the initial condition $x^i(t_o) \equiv x_o^i = 0$, $\gamma(t_o) = \gamma_o$ (see Eq. 49) can be integrated numerically. The results of this integration will then be used to evaluate the coordinates $x^i(t_e)$ of photons, at the emission time t_e , and then their redshift. From Eqs. (60), (61), (74) we find

$$1 + z_e(\theta, \varphi) = \frac{T_e}{T_o(\theta, \varphi)} = \gamma_o^{-1/2} \Xi(\theta, \varphi, t_o; t_e) \quad (79)$$

Note that the dependency on the parameters of the finite coordinate transformation is manifest in the equation Eq.(79). In the diagonal form the system of equations describing the null geodesics can be obtained from the system Eq.(32), Eqs.(76-77-78) effectively, by

chopping off the terms containing b , b_o . The observed quantity $\frac{\Delta T}{T}$ is related with the above expressions through

$$\frac{\Delta T}{T}(\theta, \varphi) = \frac{T_o/T_e}{\langle T_o/T_e \rangle} - 1, \quad (80)$$

where the mean value is understood as an integration over all the sky angles θ, φ . The quadrupole moment of the temperature field $\frac{\Delta T}{T}$ is given by

$$a_2 = \frac{5}{16\pi^{3/2}} \frac{1}{|\langle T_o/T_e \rangle|} \left[3(I_1^2 + I_2^2) + 12(I_3^2 + I_4^2) + I_5^2 \right]^{1/2}, \quad (81)$$

where

$$I_1 \equiv \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin^3 \theta \sin 2\varphi \frac{T_o}{T_e}(\theta, \varphi) \quad (82)$$

$$I_2 \equiv \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin^3 \theta \cos 2\varphi \frac{T_o}{T_e}(\theta, \varphi) \quad (83)$$

$$I_3 \equiv \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin^2 \theta \cos \theta \sin \varphi \frac{T_o}{T_e}(\theta, \varphi) \quad (84)$$

$$I_4 \equiv \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin^2 \theta \cos \theta \cos \varphi \frac{T_o}{T_e}(\theta, \varphi) \quad (85)$$

$$I_5 \equiv \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta \left(3 \cos^2 \theta - 1 \right) \frac{T_o}{T_e}(\theta, \varphi) \quad (86)$$

and

$$\langle T_o/T_e \rangle = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta \frac{T_o}{T_e}(\theta, \varphi). \quad (87)$$

The results of the numerical work done, for the general Bianchi V cosmological solution with dust, are as follows:

- The solution in the off diagonal form has five arbitrary parameters. We determine two of the five parameters, choosing values for q_o , Ω_o , H_o . Doing this we also determine the present value of the shear anisotropy $\left(\frac{\Sigma}{\Theta}\right)_{today} \equiv \left(\frac{\Sigma}{\Theta}\right)_o$ which is small if $q_o \simeq \Omega_o/2$. We are left with three free parameters k , ϑ_1^1 , ϑ_1^2 . These extra three parameters, which are not present in the diagonal dust Bianchi solution, studied in [9], determine how much the diagonal elements differ inter se as well as with the non diagonal element. The choice of different values for these extra parameters select a solution in the space

of the g.c.t. equivalent metrics. However these different choices must not change the multipole moments since they are true scalar numbers. Indeed the numerical calculation exhibits exactly this; the quadrupole moment a_2 , for the diagonal and the non-diagonal form is the same up to next to next order of magnitude. We find that the anisotropy and consequently the temperature fluctuations induced from the geometry, *are small* if the shear anisotropy $\left(\frac{\Sigma}{\Theta}\right)_o$ and the electric and magnetic scalars $\left(\frac{\sqrt{|E^{\mu\nu}E_{\mu\nu}|}}{H^2}\right)_o$, $\left(\frac{\sqrt{|H^{\mu\nu}H_{\mu\nu}|}}{H^2}\right)_o$ are extremely small. All these three scalars are attaining very small values if the value of w is close to zero.

- Compared to the previous results in [9] we found somewhat smaller temperature fluctuations induced from a dust Bianchi V solution to the CMBR for $\left(\frac{\Sigma}{\Theta}\right)_o = 10^{-9}$. We computed the temperature map and we found that, both for the solution in the diagonal form, analogous to the solution used in [9]¹, and for the solution written in the off diagonal form, the temperature allover the sky can be $T_o = 2.7277 \pm 10^{-5}$ when $\left(\frac{\Sigma}{\Theta}\right)_o = 10^{-9}$. As we have already pointed out, this numerical value does not depend on the choice of values for the three free parameters $k, \vartheta_1^1, \vartheta_1^2$.
- In solving numerically the system Eq.(32), Eqs.(76-77-78) one has to exercise great care and demand large accuracy² since otherwise large anisotropies, of non geometric origin, can appear as a result of the poor precision.
- The anisotropic distortion $\left(\frac{\Delta T}{T}\right)_{\text{rms}} = \sqrt{\frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta \left(\frac{\Delta T}{T}\right)^2}$, of the CMBR is smaller for small values of the flatness parameter Ω_o as can be seen below.

¹However the results in this work, are given for matter content, consisting of dust and radiation.

²Large working precision is required, much more than the double precision (16-digits) normally used.

	for $\left(\frac{\Sigma}{\Theta}\right)_o = 10^{-9}$				
Ω_o	0.1	0.2	0.4	0.8	0.98
$\left(\frac{\Delta T}{T}\right)_{\text{rms}}$	10^{-6}	10^{-6}	10^{-5}	10^{-5}	10^{-4}

- We found small values of a_2 coming after integrating all the sky.

	for $\left(\frac{\Sigma}{\Theta}\right)_o = 10^{-9}$				
Ω_o	0.1	0.2	0.4	0.8	0.98
$(a_2)_{\text{all the sky}}$	10^{-6}	5×10^{-6}	10^{-5}	10^{-4}	10^{-4}

- An interesting outcome of our calculations is that the solution in the off diagonal form yields a temperature map all over the sky that is different from the one resulting in the diagonal form. The dust diagonal metric, we used, gives a quadrupole like structure presented in Fig. 1, while the non diagonal metric gives a quadrupole like structure given in Fig 2. Both Figures are drawn for $\Omega_o = 0.4$ and $\left(\frac{\Sigma}{\Theta}\right)_o = 10^{-9}$. Note that the actual observed angles in the sky are shifted from the ones used in calculations $\theta_{obs} = \pi - \theta$, $\varphi_{obs} = \pi + \varphi$.

- As far as the focusing effect is concerned, known to appear in open anisotropic models [21], the depicted situation inferred from Figure 1 is in complete agreement with the findings in [9]; the distance between the focusing point and the points of the maximum anisotropy is about $\pi/8$. If one wants to reproduce the corresponding Figure in [9], from our Figure 1, one must shift properly the axes (the focusing point which is close to $\theta = \pi/2$, $\varphi = \pi/2$ should be shifted to $\theta = 0$, $\varphi = 0$). Lastly we observe that in Figure 2 one gets the same magnitude of the focusing angle.

IV. CONCLUSIONS

The exact solution of a Type V Bianchi Universe consisted of an untilted perfect fluid with a diffuse matter's equation of state is studied. A simple method for solving the null

geodesics is employed, which makes use of some appropriate inner products, constant along the geodesics.

We found that small temperature fluctuations consisted with the observational data can be achieved if the parameters of the model are such that give $\left(\frac{\Sigma}{\Theta}\right)_o \leq 10^{-9}$. This fact justifies our assessment that the model presented, can be a realistic description of our Universe at least for the epoch commencing from last scattering and reaching the present time.

The most important feature of the model studied is that the temperature pattern produced by the geometric anisotropies, repeatedly calculated in the literature, estimated after solving the null geodesics, is a strongly dependent on the spatial coordinate system used, picture. We exhibited how large this dependency is.

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FIGURES

FIG. 1. The temperature map for the diagonal solution with $\Omega_o = 0.4$. Regions with higher temperature values are lighter.

FIG. 2. The temperature map for the non diagonal solution with $\Omega_o = 0.4$.

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Figure 1

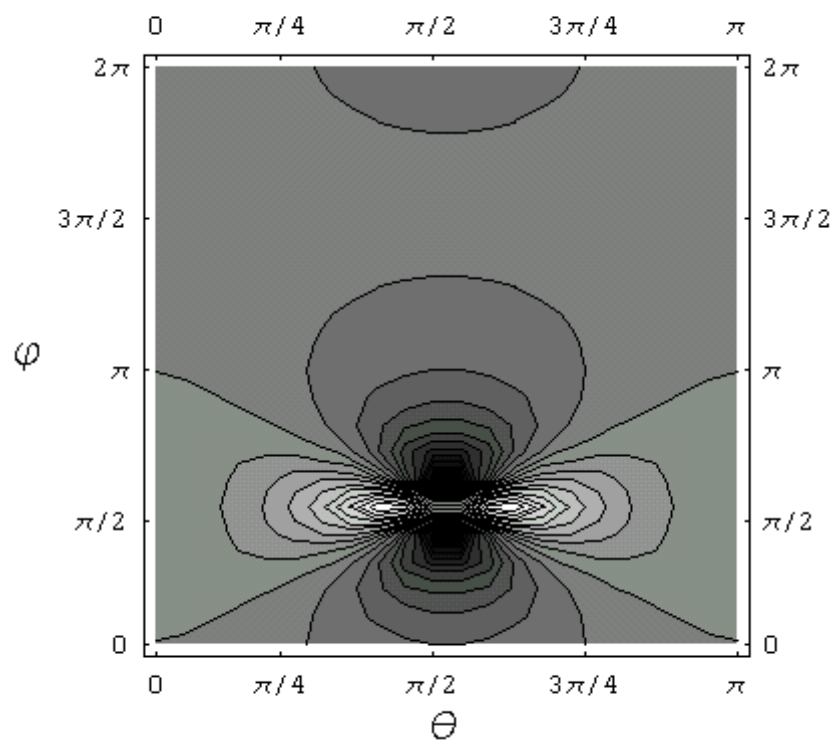


Figure 2

